B.Sc. Physics, Part-III,
Paper-V, Group-A

Legendre equation and Legendre polynomials:
(Part-I)

Consider the differential equation:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \text{———} \quad (1)$$

, $n \to$ real number.

Eq. (1) is called the Legendre's equation which is the special case of generalized Legendre equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0 \quad \text{———} \quad (2)$$

for $m^2 = 0$, Eq. (2) reduces to Eq. (1). The solution of the Legendre equation is called the associated Legendre functions. Below we discuss solution of Eq. (1)

The solution will be assumed in the form of power series given as

$$y = a_0 x^j + a_1 x^{j+1} + a_2 x^{j+2} + \cdots \quad \text{for } a_0 \neq 0 \quad \text{———} \quad (3)$$

Substituting (3) in (1), we obtain

$$a_0 (j)(j-1)x^{j-2} + a_1 (j+1) j x^{j+1} + \cdots + [a_{r+2}(j+r+2)(j+r+1)$$

$$- \{(j+r)(j+r+1) - n(n+1)\}a_r]\,x^{j+r} + \cdots = 0 \quad \text{———} \quad (4)$$

In this ~~eq.~~ expansion the coefficient of each power of x must vanish separately. Equating to zero the coefficient of lowest power of x.

$$a_0 j(j-1) = 0, \quad \Rightarrow j = 0, 1 \quad \text{———} \quad (5)$$

Next, equating to zero the coefficients of x^{J-1} and x^{J+r}, we obtain

$$a_1 (J+1) J = 0 \quad \text{——— (6)}$$

$$a_{r+2} (J+r+2)(J+r+1) - \{(J+r)(J+r+1) - n(n+1)\} a_r = 0 \quad \text{——— (7)}$$

When $J=0$; (2) is satisfied and thus $a_1 \neq 0$.

Eq. (7) gives for $r = 0, 1, 2, ---$

$$a_2 = -\frac{n(n+1)}{2!} a_0$$

$$a_3 = -\frac{(n-1)(n+2)}{3!} a_1$$

$$a_4 = -\frac{(n-2)(n+3)}{4 \times 3} a_2 = \frac{n(n-2)(n+1)(n+3)}{4!}$$

$$a_5 = -\frac{(n-3)(n+4)}{5 \times 4} a_3 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!} a_1$$

$\vdots$

Thus for $J=0$, we obtain the following two independent solution of eq (1)

$$y_1 = a_0 \left\{ 1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)\, n\, (n+1)(n+3)}{4!} x^4 - \dots \right\} \quad \text{——— (8)}$$

$$y_2 = a_1 \left\{ x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 \dots \right. \quad \text{——— (9)}$$

Again for $J=1$, we see from (6) that $a_1 = 0$, therefore eq. (9) will be the solution.

$y = y_1 + y_2$ is the general solution of eq. (1).